Differential Equations

Differential Equations And Their Classification

An equation that involves differentials or derivatives is known as a differential equation.

Some examples of such equations are

$$x\,dx + y\,dy = 0 \quad \text{---(1)}$$

$$\frac{dy}{dx} + 3y = x \quad \text{---(2)}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x \quad \text{---(3)}$$

$$\frac{d^ny}{dx^n} + \frac{dy}{dx} = e^{3x} \quad \text{---(4)}$$

$$\left(\frac{dy}{dx}\right)^2 = x^2 + \sqrt{x} \quad \text{---(5)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{---(6)}$$

When an equation contains derivatives with respect to a single independent variable, it is called an ordinary differential equation. In case of more than one independent variables involved in the problem, the derivatives occuring are partial and the eqn. is called a partial differential eqn.

Only eqn (6) is the partial differential eqn.; all others are ordinary differential eqn.

The order of the highest ordered derivative occurring in it is known as the order of the differential eqn. The degree of a differential eqn is the highest integral ~~power~~ power of the derivative that determines the order of the equation.

Equations ~~of~~ (1), (2) above are of first order and first degree; eqn (5) is of first order and second degree; eqn (3) is of 2nd order and 1st degree.

Formation of Differential Eqn

For every real value of m, $y = mx$ represents a straight line through (0,0); taking all possible real values of m, one gets all possible straight lines through the origin.

$$y = mx \quad \text{---}(1)$$

is said to represent the family of straight lines through the origin where m is an arbitrary constant.

Differentiating (1) w.r.to x,

$$\frac{dy}{dx} = m \quad \text{----}(2)$$

Eliminating m betn (1) & (2), one gets

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{----}(3)$$

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which is free from the arbitrary constant m and is the differential eqn for the family.

As a rule, one requires $n+1$ relations to eliminate n quantities. If a relation

$$f(x, y, c_1, c_2, \ldots c_n) = 0 \quad \text{—— (4)}$$

containing n arbitrary constants is given, it is differentiated successively n times w.r.to x to get n new relation. These together with the given relatn (4) constitute a set of $n+1$ relations among which the n constants $c_1, c_2, \ldots c_n$ can be eliminated. The resulting eqn will evidently contain derivatives upto nth order and, therefore, be a differential eqn of nth order.

Exa: 1

$$\text{Let } y = A \sin x \quad \text{—— (1)}$$

be a relation where A is an arbitrary constant. To eliminate A another relation is required.

Differentiating (1) w.r.to x,

$$\frac{dy}{dx} = A\cos x \quad \text{—— (2)}$$

Elimination of A betn (1) & (2) yields

$$\frac{dy}{dx} = y \cot x \quad \text{—— (3)}$$

which is a differential eqn of order one.

Ex: 2

$y = Ae^{x} + Be^{-x}$ ——— (1)

is an eqn. with two arbitrary constants A & B. Two more relations are required to eliminate them.

Differentiating (1) successively twice w.r.to x

$$\frac{dy}{dx} = Ae^{x} - Be^{-x} \quad \text{—— (2)}$$

$$\frac{d^2y}{dx^2} = Ae^{x} + Be^{-x} \quad \text{—— (3)}$$

Comparing (1) & (3),

$$\frac{d^2y}{dx^2} = y \quad \text{——— (4)}$$

which is free from the arbitrary constants and is a differential eqn of order two.

Soln of a Differential Eqn:

A relation like $y = f(x)$ or $f(x, y) = 0$ betn the varibles is called a solution of a differential eqn. if it reduces the eqn. to an identity when substituted into it.

For example, $y = e^{x}$ is a solution of the eqn.

$$\frac{dy}{dx} = y \quad \text{——— (a)}$$

since taking $y = e^{x}$ and so $\frac{dy}{dx} = e^{x}$, (a) is reduced to $e^{x} = e^{x}$ which is identically true. But in this case $y = 2e^{x}$, $y = 7e^{x}$, $4e^{x}$ are each a solⁿ. of (a). In fact, $y = ce^{x}$ is a solⁿ for any constant value of c. Here $y = ce^{x}$ is known as the <u>general solution</u> of the eqⁿ. (a), $y = 2e^{x}$, $y = 7e^{x}$ are known as <u>particular solutions</u>. Particular solⁿ.s are obtained from the general solⁿ. by taking particular value(s) of the arbitrary constant(s).

It is not always possible to solve a given differential eqⁿ. Only two of these types, one of first order and the other of 2nd order, will be discussed in this section. They are

(i) $\frac{dy}{dx} = f(x)\, g(y)$

(ii) $\frac{d^2y}{dx^2} = h(x)$

<u>Type I</u>

The eqⁿ.

$$\frac{dy}{dx} = f(x) \quad \text{---(1)}$$

can be written in the form $dy = f(x)dx$.

Integrating both sides one gets

$$y = \int f(x)\, dx + C \quad \text{---(2)}$$

as the general solⁿ of (1). Similarly, the eqⁿ.

$$\frac{dy}{dx} = g(y) \quad \text{---(3)}$$

can be written in the form $\frac{dy}{g(y)} = dx$.

Integrating, $\int \frac{dy}{g(y)} = x + C \quad \text{--(4)}$

is obtained as the general solⁿ of (3). But both of these forms are special cases of the eqⁿ.

$$\frac{dy}{dx} = f(x)\, g(y) \quad \text{---} (5)$$

which can be written in the form

$$\frac{dy}{g(y)} = f(x)\, dx \quad \text{---} (6)$$

This process of collecting all functions of x with dx and all functions of y with dy is known as the process of <u>separation of variables</u>. Integrating both sides of (6)

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C \quad \text{---} (7)$$

as the general solution.

<u>Exa:</u>

Solve $\frac{dy}{dx} = x^2 + 2x + 5$

<u>Solⁿ.</u>

The given eqⁿ can be written in the form

$$dy = (x^2 + 2x + 5)$$

Integrating both sides

$$\int dy = \int (x^2 + 2x + 5)\, dx$$

or $$y = \frac{x^3}{3} + x^2 + 5x + C$$

This is the general solution.

Type II :

The second order eqn.

$$\frac{d^2y}{dx^2} = h(x) \quad \text{--- (1)}$$

can be solved in two steps each time a first order eqn. being solved. In the Ist step, taking $\frac{dy}{dx} = p$ and so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dp}{dx}$$

the eqn. (1) becomes $\frac{dp}{dx} = h(x)$, On integration

$$p = \int h(x)\,dx + A$$

$$\text{or } \frac{dy}{dx} = \int h(x)\,dx + A \quad \text{--- (2)}$$

$$= \theta(x) + A \quad \text{where } \theta(x) = \int h(x)\,dx$$

This is called a first integral or an intermediate integral. Eqn (2 can be integrated again to yield the final solution.

Thy

$$\int dy = \int [\theta(x) + A]\,dx$$

$$\text{or } y = \int \theta(x)\,dx + Ax + B \quad \text{--- (3)}$$

which is the general soln. containing two arbitrary constants A & B.

Exa:

Solve $\frac{d^2y}{dx^2} = 6x + 2$

Soln. Taking $\frac{dy}{dx} = p$, the eqn. becomes

$$\frac{dp}{dx} = 6x + 2$$

On integration, $\int dp = \int (6x+2)\,dx$

or $p = \frac{dy}{dx} = 3x^2 + 2x + A$

Integrating again, $\int dy = \int (3x^2 + 2x + A)\,dx$

or $y = x^3 + x^2 + Ax + B$

which is the general sol^n.

14.4 PARTICULAR SOLUTION :

The general solution of a differential equation contains as many arbitrary constants as the order of the equation. Particular solutions are obtained by putting particular values for these constants. Sometimes, some conditions are required to be fulfilled by the solutions, often because of the need of physical situations. Usually the number of such conditions is equal to the order of the equation (and so to the number of arbitrary constants). The particular values for the arbitrary constants are determined so as to satisfy the given conditions.

Example 10 :

Solve the equation $\frac{dy}{dx} = \cos x$ subject to the condition, $y = 2$ when $x = 0$

Solution :

The equation is $dy = \cos x dx$

Integrating both sides,

$$y = \sin x + C$$

This is the general solution. Putting the given condition.

$$2 = \sin 0 + C$$

or $C = 2$

Hence the particular solution that will satisfy the given condition is $y = \sin x + 2$

Example 11 :

Find the particular solution of the equation

$$\frac{d^2y}{dx^2} = 2x, \text{ given that when } x = 0, y = 2 \text{ and } \frac{dy}{dx} = 3$$

Solution :

The equation is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2x$$

On integration, $\frac{dy}{dx} = x^2 + A$ ... (1)

On further integration, $y = \frac{x^3}{3} + Ax + B$ ... (2)

This is the general solution containing two arbitrary constants A and B.

Using the condition, $\frac{dy}{dx} = 3$ when $x = 0$, from (1) we get

$$3 = 0 + A \Rightarrow A = 3$$

Using the condition, $y = 2$ when $x = 0$, we get from (2)

$$2 = 0 + 0 + B \Rightarrow B = 2$$

So the required particular solution is $y = \frac{1}{3}x^3 + 3x + 2$.

Determine the order and degree of each of the following differential equations.

(i) $y \sec^2 x \, dx + \tan x \, dy = 0$

(ii) $\left(\frac{dy}{dx}\right)^4 + y^5 = \frac{d^3y}{dx^3}$

(iii) $a\dfrac{d^2y}{dx^2} = \left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

(iv) $\tan^{-1}\sqrt{\dfrac{dy}{dx}} = x$

(v) $ln\left(\dfrac{d^2y}{dx^2}\right) = y$

(vi) $\dfrac{\dfrac{dy}{dt}}{y+\dfrac{dy}{dt}} = \dfrac{yt}{\dfrac{dy}{dt}}$

(vii) $\dfrac{d^2y}{du^2} = \dfrac{3y+\dfrac{dy}{du}}{\sqrt{\dfrac{d^2y}{du^2}}}$

(viii) $e^{\frac{dy}{dx}} = x^2$

14.5 Linear Differential Equations

A differential equation is said to be **linear** if the dependent variable and its differential co-efficients occuring in the eqation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \qquad .....(1)$$

where P, Q are functions of x.

In this section we shall be concerned with linear differential equations of first order only.

The standard technique to slolve linear equations of the form (1) is to multiply both sides with $e^{\int Pdx}$ After multiplication, we get

$$e^{\int Pdx}\frac{dy}{dx} + \left(Pe^{\int Pdx}\right)y = Q\,e^{\int Pdx} \qquad .... (2)$$

If may be easily seen that the left hand side of (2) is the derivative of the product $ye^{\int Pdx}$ with respect to x and the right hand side is a function of x alone.

So we can write (2) as

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Q\,e^{\int Pdx} \qquad ....(3)$$

Integrating both sides of (3) with respect to x, we get

$$ye^{\int Pdx} = \int Q\,e^{\int Pdx}dx + C,$$

where C is an arbitary constant.

So $$y = e^{-\int Pdx}\left(\int Q\,e^{\int Pdx}dx + C\right) \qquad .... (4)$$

is the general solution of the differential equation (1).

Note : The factor $e^{\int Pdx}$ on multiplication with the left hand side of (1) reduces it to an exact differential and is called the **integrating factor** of the differential equation (1).

We summarize below the steps involved is solving a first order linear differential equation.

(a) Write the equation in the standard form :

$$\frac{dy}{dx} + Py = Q$$

(b) Determine the integrating factor $e^{\int Pdx}$.

(c) Multiply both sides of the equation with $e^{\int Pdx}$ when the equation is in standard form.

(d) Integrate the resulting equation to obtain the required solution.

Note : (1) There are differential equations which are not linear in y but linear in x. Such equations in standard form look like

$$\frac{dx}{dy} + Px = Q$$

Where P, Q are functions of y alone.

In this case the integrating factor is $e^{\int Pdy}$

(2) A differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n, n \neq 1 \quad \text{.... (5)}$$

where P, Q are functions of x is called **Bernoulli's equation**.

This equation can be reduced to the

form $\frac{dz}{dx} + (1-n)Pz = (1-n)Q$ which is linear with z as dependent variable, by putting $z = y^{-n+1}$.

Example 12 : Solve : $(1+x^2)\frac{dy}{dx} + 2xy - x^3 = 0$

Solution : $(1+x^2)\frac{dy}{dx} + 2xy - x^3 = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x^3}{1+x^2}$$

which is a linear differential equation of first order.

Here $P = \frac{2x}{1+x^2}$ and $Q = \frac{x^3}{1+x^2}$

So the integrating factor $= e^{\int Pdx}$

$$= e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)} = 1 + x^2.$$

Multiplying both sides of the equation with $1 + x^2$ we get

$$(1+x^2)\frac{dy}{dx} + 2xy = x^3$$

i.e., $$\frac{d}{dx}\left\{(1+x^2)y\right\} = x^3 \quad \text{... (2)}$$

So $(1 + x^2)\,y = \int x^3\,dx + c$

$$= \frac{x^4}{4} + c,$$

where c is an arbitorary constant. Hence the general solution of the given differential equation is given by

$$y = \frac{x^4}{4(1+x^2)} + \frac{c}{1+x^2}$$

Example 13 : Solve : $(1 + y^2)\,dx + x\,dy = \tan^{-1}y\,dy$

Solution : $(1 + y^2)dx + xdy = \tan^{-1}y\,dy$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2} \quad \text{... (1)}$$

which is a linear differential equation of first order.

Here $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

So the integrating factor $= e^{\int Pdy}$

$$= e^{\int \frac{dy}{1+y^2}}$$

$$= e^{\tan^{-1}y}$$

Multiplying both sides of the equation (1) with $e^{\tan^{-1}y}$, we get

$$e^{\tan^{-1}y}\frac{dx}{dy}+\frac{e^{\tan^{-1}y}}{1+y^2}x=e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}$$

i.e. $$\frac{d}{dy}\left(x\,e^{\tan^{-1}y}\right)=e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2} \qquad \text{... (2)}$$

Integrating both sides of (2) with respect to y, we get

$$x\,e^{\tan^{-1}y}=\int e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}\,dy+c$$

$$=\int te^{t}dt + c, \text{ where } t = \tan^{-1}y$$

$$= e^{t}(t-1)+c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y-1)+c.$$

So $x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$ ... (3)

where c is an arbitrary constant.

(3) is the general solution of the given differential equation.

Solve the following differential equations :

1. $\frac{dy}{dx} + y = e^{-x}$

2. $(x^2 - 1) \frac{dy}{dx} + 2xy = 1$

3. $(1 - x^2) \frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

4. $x \log x \frac{dy}{dx} + y = 2 \log x$

5. $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$

dy